Let's see if we can approximate a continuous function
\[ f : X \to Y \] with a linear function
\[ Y \leftarrow X \ \text{with} \ \lambda \in \text{Lin}(X,Y). \]

For this to be meaningful, \( X, Y \) should have linear structure as well as a topology. We'll assume that \( X \) and \( Y \) are real, finite-dimensional "topological vector spaces." This is just a Hausdoff space where \( (x,x') \to x + x', (a,x) \to ax \) and \( x \to -x \) are continuous. Geroch proves that any such space is isomorph to the euclidean topology on \( \mathbb{R}^n \) given by the metric
\[ d(x,x') = |x - x'| \]
where \( d : X \to \mathbb{R} \) is a norm, i.e.
\[ |ax| = |a||x| \quad a \in \mathbb{R} \]
\[ |x + x'| \leq |x| + |x'| \]
\[ |x| > 0 \quad |x| = 0 \Rightarrow x = 0. \]

Exercise: Prove that \( d \), as defined, is a metric.

We can now use the norm to measure how close a linear map is to a continuous map.

Aside: Is \( \mathbb{R}^n \) boring? How about this: Are all manifolds as topological spaces \( \mathbb{R}^n \) isomorphic?

Answer: Yes, except for \( n = 4 \) when there are an infinite number of solutions. Donaldson, 1986 Fields Medal.
\[
X \xrightarrow{f} Y \quad f \in \mathcal{M}(X, Y)
\]
\[
X \xrightarrow{l} Y \quad l \in \text{Lin}(X; Y)
\]

For a fixed \(x \in X\), let \(R\) be defined by
\[
f(x + h) = f(x) + L(h) + R(x, h)
\]
If \(h \mapsto \begin{cases} 1R(x, h)/|h| & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}\) is continuous,
then \(f\) is said to be \underline{differentiable} at \(x\). In this case, \(L\) is called the \underline{differential} of \(f\) at \(x\), often denoted \(df_x \in \text{Lin}(X; Y)\).

The differentials have various nice properties, e.g.

**Theorem:** Differentials are unique.

**Proof.** Suppose that \(L\) and \(L'\) are both differentials of \(f\) at \(x\) and \(L\) and \(L'\) differ by differ at some nonzero \(h \in X\).

Then
\[
\frac{L(h) - L'(h)}{|h|} = \frac{R(x, h) - R(x, h)}{|h|} = \frac{R(x, h)}{|h|}
\]
if \(h \neq 0\)

\[
(L - L')(h/|h|) = \left| \frac{R(x, h)}{|h|} \right| \leq \frac{|R'(x, h)|}{|h|} + \frac{|R(x, h)|}{|h|}.
\]

Since the quantity on the left is constant on the line \(X \times 0 \times \mathbb{R}^t\), any neighborhood of \(0\) in \(X\) contains such a point (why?) and so \(h \mapsto \begin{cases} 1R'(x, h)/|h| + |R(x, h)/|h| & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}\) continuous \(\Rightarrow L = L'\) are not differentials of \(f\) at \(x\) \(\Rightarrow L = L'\).

**Exercise:** Fill in the details of this proof.
Besides uniqueness, the most important fact about differentials is

**Theorem (The chain rule):** If \( X \overset{f}{\to} Y \) and \( Y \overset{g}{\to} Z \) are differentiable, then \( X \overset{g \circ f}{\to} Z \) is differentiable and, at any \( x \in X \),
\[
d (g \circ f)_x = dg_{g(f(x))} \circ df_x.
\]

**Proof:** Given \( x \in X \), let \( f(x + h) = f(x) + df_x(h) + R_f(x, h) \). Then \( g(f(x + h)) = g(f(x)) + dg_{f(x)} \circ df_x(h) + R_g(f(x), df_x(h) + R_f(x, h)) \) where \( R(x, h) = dg_{f(x)}(R_f(x, h)) + R_g(f(x), df_x(h) + R_f(x, h)) \).

Since \( h \mapsto (R(x, h))/h \) if \( h \neq 0 \) is continuous, we're done.

**Exercise:** Fill in the details of the proof.

Notice that these two results give us a functor

\[
(X, x) \overset{f}{\to} (Y, y) \quad \text{Pointed topological spaces}
\]

\[
T(X, x) \overset{Tf}{\to} T(Y, y) \quad \text{Real vector spaces, } Tf = df_x
\]

\( T \) is called the "Tangent space" functor.

Notice how the functional equation \( T(g \circ f) = Tg \circ Tf \)

is exactly the chain rule.
It's often useful to look at the differential as a "section" of a fiber bundle.

Let \( \mathcal{E} = \{ (x, v) : x \in X, v \in T(x, x) \} \)

This is the "tangent bundle" with projection \( \pi : \mathcal{E} \to X, \pi : (x, v) \mapsto x \).

Then, the differential of \( f \) is a particular section of \( \mathcal{E} \), i.e. a \( \pi^d = i_x \) i.e. a choice of \( v \in T(x, x) \) for every \( x \in X \).

In general, the category of bundles over \( X \) is:

**Objects:** \( E \to X, F \to X \), etc.

**Morphisms:** \( (E, \pi) \to (F, \nu) \) s.t. \( \pi \downarrow \nu \) commutes.

Bundles are a convenient way to define, e.g., vector fields. A vector field on manifold \( M \) is a section of \( TM \to M \)

i.e. for each \( p \in M \), a vector

\[
\nu(p) = a_1(p)e_1^p + a_2(p)e_2^p + \ldots + a_n(p)e_n^p
\]

where \( \{ e_1^p, e_2^p, \ldots, e_n^p \} \) are a basis of \( T(M, p) \equiv T_p(M) \).
Differential Forms

Let us consider differentiable functions \( U \to \mathbb{R} \) where \( U \) is an open subset of \( \mathbb{R}^n \). The simplest such functions are projections

\[
x_i: (a_1, a_2, \ldots, a_i, \ldots, a_n) \mapsto a_i
\]

Since \( x_i(x+h) - x_i(x) + x_i(h) + o(h) \), \( x_i \) is differentiable with \( dx_i: h \mapsto h_i \). Notice that \( dx_i \in \text{Lin}(\mathbb{R}^n; \mathbb{R}) = (\mathbb{R}^n)^* \) and \( dx_1, dx_2, \ldots, dx_n \) are a basis for \( (\mathbb{R}^n)^* \).

Thus, in general,

\[
df_x(x) = \sum_{i=1}^{m} a_i(x) \, dx_i(x)
\]

The \( a_i(x) \) are called partial derivatives, usually written \( a_i(x) = \frac{\partial f}{\partial x_i}(x) \).

Exercise: Consider \( f(x + \lambda e_i) \) and show that \( a_i(x) \) is the partial derivative of \( f \) defined in the usual way.

Given any vector space \( V \), (e.g. \( (\mathbb{R}^n)^* \)), we have already defined \( V \wedge V, V \wedge V \wedge V \), etc., meaning that \( dx \wedge dy + dy \wedge dz + dz \wedge dx \in \wedge^3 V \), etc., are defined.

We would like to extend calculus to these geometric objects as well.
If we define a "k-form" $I = (i_1, i_2, ..., i_k)$ to be an increasing list of $k$ integers between 1 and $n$, then

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$$

is a basis of $\Lambda^k(\mathbb{R}^n)^\times$. Thus, the general $k$-form can be written

$$\omega = \sum_I a_I \, dx_I \quad a_I \in \mathbb{R}$$

note that $a_I$ is a real constant. We can extend this to $U \subset \mathbb{R}^n$ by letting

$$\omega(x) = \sum_I a_I(x) \, dx_I$$

where $a_I(x)$ are smooth functions from $U$ to $\mathbb{R}$. In this context, $\omega$ is called a differentiable $k$-form.

Notice that differentiable 0-forms are simply real-valued functions on $U$. This means that we already have

$$\Lambda^0(\mathbb{R}^n)^\times \xrightarrow{d} \Lambda^1(\mathbb{R}^n)^\times$$

i.e. the differential. We can extend this to

$$\Lambda^k(\mathbb{R}^n)^\times \xrightarrow{d} \Lambda^{k+1}(\mathbb{R}^n)^\times$$

by

$$d\omega = \sum_I d^I a_I \wedge dx_I$$

d is called the "exterior derivative."
The exterior derivative has these properties:

a) If $f \in \Lambda^0(\mathbb{R}^n)\star$, $d(f)$ is the differential.

b) $d(w_1 + w_2) = dw_1 + dw_2$

c) $d(dw) = 0$

d) $d(w \wedge y) = dw \wedge y + (-1)^k wy \wedge dw$ where $w$ is a $k$-form and $y$ is an $s$-form.

Exercise. Prove (a), (b), (c), (d) above.

To prove (c), let $f: \mathbb{R}^n \to \mathbb{R}$ be a $0$-form. Then

$$d(df) = d\left( \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \right) = \sum_{j=1}^{n} d\left( \frac{\partial f}{\partial x_j} \right) \wedge dx_j$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} dx_k \wedge dx_j = 0$$

since $\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial f}{\partial x_j} \wedge \frac{\partial f}{\partial x_k}$ for smooth $f$. Using (d), this is true in any $k$-form.

If $dw = 0$, $w$ is called closed, i.e., $w \in \ker d$.

If $w = dy$ for some form $y$, $w$ is called exact, i.e. $w \in \text{Im } d$.

Since $d^2 = 0$, every exact form is closed.

Theorem (Poincaré): If $U \subset \mathbb{R}^n$ is contractible, then a $k$-form on $U$ is closed if and only if it is exact.
Example: \( \mathbb{R}^3 \)

For \( f \in \Lambda^3(\mathbb{R}^3)^* \),

\[
\frac{df}{dx} = \frac{2f}{\partial x} dx + \frac{2f}{\partial y} dy + \frac{2f}{\partial z} dz
\]

For \( f_1 dx + f_2 dy + f_3 dz \in \Lambda^1(\mathbb{R}^3)^* \)

\[
d(f_1 dx + f_2 dy + f_3 dz) =
\left( \frac{2f_1}{\partial y} - \frac{2f_1}{\partial x} \right) dy \wedge dx - \left( \frac{2f_2}{\partial z} - \frac{2f_2}{\partial x} \right) dx \wedge dz + \left( \frac{2f_3}{\partial y} - \frac{2f_3}{\partial y} \right) dy \wedge dz
\]

For \( \omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy \in \Lambda^2(\mathbb{R}^3)^* \)

\[
d\omega = \frac{2f_1}{\partial x} dx \wedge dy \wedge dz + \frac{2f_2}{\partial y} dy \wedge dz \wedge dx + \frac{2f_3}{\partial z} dz \wedge dx \wedge dy
\]

\[- \left( \frac{2f_1}{\partial x} + \frac{2f_2}{\partial y} + \frac{2f_3}{\partial z} \right) dx \wedge dy \wedge dz \]

\[
d(0\text{-form}) \equiv \text{Gradient}
\]

\[
d(1\text{-form}) \equiv \text{Curl}
\]

\[
d(2\text{-form}) \equiv \text{Divergence}
\]
Unlike tensor fields, differential forms can be pulled back across smooth maps.

\[ \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \quad f \text{ smooth} \]

If we have a k-form \( \omega \) on \( \mathbb{R}^m \), we can define

\[ \omega = \sum_I a_I \, dy_{i_1} \wedge dy_{i_2} \wedge \ldots \wedge dy_{i_k} \]

\[ f^* \omega = \sum_I (a_I \circ f) \, d(y_{i_1} \circ f) \wedge d(y_{i_2} \circ f) \wedge \ldots \wedge d(y_{i_k} \circ f) \]

Pullbacks have these properties:

1. \( f^* (\omega + \omega') = f^* \omega + f^* \omega' \)
2. \( f^* (g \cdot \omega) = f^* (g) \cdot f^* (\omega) \)
3. \( f^* (\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k) = f^* (\omega_1) \wedge \ldots \wedge f^* (\omega_k) \)

for k-forms \( \omega \).

They also interact nicely with the exterior derivative

\[ d(f^* \omega) = f^* (d \omega) \]

\[ f^* f^* \text{ corresponds to} \]
Example. Let $U = \{ (r, \theta) : r > 0, 0 < \theta < 2\pi \} \subset \mathbb{R}^2$

$y : U \to \mathbb{R}^2$ be $y : (r, \theta) \mapsto (r \cos \theta, r \sin \theta)$.

Then the pullback of the volume form $dx \wedge dy$ is given by

$y^*(dx \wedge dy) = d(x \circ y) \wedge d(y \circ y) = r \, dr \wedge d\theta$

Exercise: Fill in the steps.

Exercise: Let $w = \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy$ be a 1-form in $\mathbb{R}^2 - \{0\}$. Show that $y^*w = d\theta$.

Exercise: Let $f : (r, \theta, y) \mapsto (r \sin \theta \cos y, r \sin \theta \sin y, r \cos \theta)$.

Show that $f^*(dx \wedge dy \wedge dz) = r^2 \sin \theta \, dr \wedge d\theta \wedge dy$.

Notice that the pullback of an isomorphism is "change of variables".
The sequence
\[ \Lambda^0(R^n)^\pi \xrightarrow{d} \Lambda^1(R^n)^\pi \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^n(R^n)^\pi \]
is called "the de Rham complex." \( \Lambda^*(R^n)^\pi = \bigoplus_K \Lambda^k(R^n)^\pi \)
The direct sum of vector spaces \( H^k = K d / \text{Im } d \)
\( \bigoplus_K H^k \) is called "the de Rham cohomology."

de Rham showed that this is a topological invariant.

More generally, a direct sum \( \bigoplus_K V^k \) of vector spaces with
\[ V^0 \xrightarrow{d} V^1 \xrightarrow{d} V^2 \xrightarrow{d} \ldots \]
\( d^2 = 0 \), is called a differential complex. \( \bigoplus_K \text{Ker } d / \text{Im } d \)
is called the cohomology of \( V \). A morphism between
differential complexes \( A \) and \( B \) is defined to be a
sequence of morphisms \( f : A^k \rightarrow B^k \) s.t.
\[ A^0 \rightarrow A^1 \xrightarrow{d} A^2 \xrightarrow{d} \ldots \]
\( f \downarrow \quad \downarrow f \quad \downarrow f \quad \downarrow f \)
\[ B^0 \xrightarrow{d} B^1 \xrightarrow{d} B^2 \xrightarrow{d} \ldots \]
commutes. In this language, \( \Lambda^* \) is a cofunctor from
the category of Euclidean space \( R^n \) to the category
of differential complexes
\[ R^n \xrightarrow{f} R^n \quad \Lambda^* R^n \xleftarrow{f^*} \Lambda^* R^n \]
where \( f^* \) is the pullback.
Hodge Star

An inner product \( \langle \cdot, \cdot \rangle : V \times V \to \mathbb{R} \) on a real finite dimensional vector space \( V \) is a bilinear symmetric product which is "nondegenerate" meaning that
\[
\langle v, x \rangle = 0 \text{ for all } x \in V \implies v = 0.
\]

Lemma. For \( f \in V^* \), \( f = (v \mapsto \langle v, u \rangle) \) for some unique \( u \in V \).

Proof. \( \Phi : \otimes x \mapsto (v \mapsto \langle x, v \rangle) \) from \( V \to V^* \) is one-to-one (using the nondegenerate property) \( \implies \Phi \) is an isomorphism.

Let \( \dim(V) = n \), \( V \wedge V \wedge \ldots \wedge V \cong \Lambda^n \). Fix \( \lambda \in \Lambda^k \).

Then the mapping
\[
M \mapsto \lambda \wedge M \text{ from } \Lambda^{n-k} \to \Lambda^n \in \mathbb{R}
\]
is
\[
\lambda \wedge M = \langle \nu, M \rangle 0 \text{ for some fixed basis vector } \nu \in \Lambda^n
\]
for some special unique \( \nu \in \Lambda^{n-k} \). This \( \nu \) is denoted \( \ast \lambda \), the "Hodge Star" of \( \lambda \). Notice that \( \ast \lambda \) depends both on the inner product and on the choice of basis \( \nu \in \Lambda^n \).
For example, let $m = 3$, $e_1$, $e_2$, $e_3$ be an orthonormal basis in $\mathbb{R}^3$, $\langle \cdot, \cdot \rangle$ the standard Euclidean inner product, let $d = e_1 \wedge e_2 \wedge e_3$.

Then $e_x \wedge e_y \langle \frac{e_1}{3}, e_i, e_i \rangle = \langle * (e_x \wedge e_3), \frac{e_1}{3}, e_i, e_i \rangle e_x \wedge e_y \wedge e_2$

is solved by $* (e_x \wedge e_3) = e_2$.

Similarly,

$\begin{align*}
* e_1 &= e_2 \wedge e_3, \\
* e_2 &= e_3 \wedge e_1, \\
* e_3 &= e_1 \wedge e_2
\end{align*}$

Exercise: Show that $\hat{A} \times \hat{B} = * (A \wedge B)$.

The Hodge star has this property: $\alpha \wedge * \beta = \sum \beta \wedge * \alpha$.

Exercise: For $f : \mathbb{R}^n \to \mathbb{R}$, show that $d \times (df) = \Delta f \mu$ where $\mu$ is the volume form and $\Delta$ is the Laplacian.

Example. With $F_{\mu \nu}$ the EM field strength, $F = \frac{1}{2} F_{\mu \nu} dx_\mu \wedge dx_\nu$ is the electromagnetic field. Maxwell's equations are

\[
\begin{align*}
\nabla F &= 0 \\
\text{curl } F &= 0
\end{align*}
\]

$F$ is closed. If it is defined on a contractible subset of $\mathbb{R}^4$, then $F = dA$ by Poincare's lemma.